This is going to be my first talk in our seminar series. I feel honoured that I've been given the task to initiate this series, even though there are people in this room who are a lot better than I am. As a punishment, I ask you to make me speechless and dumb with a mountain of questions! I won't mind; we're all here to learn and that should be our objective.

Today, I'd like to talk about the differences between intuitionistic logic and classical logic. Once I've highligthed some differences, I'll move on to consider its importance. With that, I'll make my mark in our seminar series on mathematical modelling. My talk is basically about the foundations of mathematics from which we can model some parts of reality. Let me warn you; my talk does not have much of mathematical modelling in mind. My aim is to give a different framework for reasoning about the real world. Of course I'll discuss some examples to make this point clearer.

Why is this talk important? Practically, it is not as important as Fuzzy logic is. Fuzzy logic deals with a world as it is – grey. In fact, fuzzy logic was developed from an application point of view. Mind you, there are numerous! This is because black and white world of mathematics becomes hazy in the real world but the classical world of mathematics remains unnerved to what's happening around. The classical world of mathematics is also unnerved about the shocks of intuitionistic logic because this logic can be embedded in classical logic, as we will see. Hence, it is not true that the proposition of an alternative framework for doing mathematics is an indication of a shaky foundation of mathematics.

This talk is important because for some people who call themselves intuitionists, the word "there exists" has been nerving. During the turn of the last century, Mathematics was going through a foundational crisis. The well-known Barber's paradox was something that erupted in that era. It was during this era that intuitionistic logic was introduced. In a sense, it has created only more problems for classical logic in a sense that reasoning has become only more longer. In a sense, intuitionistic logic is a restriction (or generalisation, if you will) of classical logic. That's how Intuitionistic logic is viewed normally. This also means that mathematics hasn't received a huge shudder with intuitionistic logic and most of mathematics remains the same. In fact, research in intuitionistic logic is almost dead. Some research has been going on with additional axioms being added, some being taken away. These are variants of intuitionistic logic, which are all under the umbrella of constructive logic. Again, research in these fields is pursued by not even a handful. So, why is this talk important from an application point of view? It's not. It's just an alternative way to view the world of mathematics and hence a different approach to model the world. This alternative approach also highlights some of the successes it has had which classical logic hasn't. I'll be pointing to a few successes of intuitionistic logic to see why.

To begin with, I won't define classical logic or go into much of its details. I will, however, be rigorous about it so you can see how the rigour for intuitionistic logic came about. I won't start with the motivation for classical logic but, of course, I'll have to do that for intuitionistic logic. However, to be fair, I'll give you the three laws of thought. The first law of identity states that p is p and not $\backsim p$. The principle of non-contradiction says that either p holds or $\backsim p$ holds. The third is the law of excluded middle, $p \vee \backsim p$. These are the laws of thought in classical logic and we're all very familiar with it. Aristotle first proposed these laws. Classical logic, based on the antiquity of the idea from the Greeks, says that ideas reside in a Platonic heaven and that they are real. We're all probably familiar that Aristotle was Plato's student and I'm not sure if the Platonic heaven in mathematics was coined after Aristotle's laws of thought without coincidence. Anyway, the "images" we have in this world are all shadows of the Platonic heaven – that's the idea of a Platonic heaven. In the real world, the idea "Man" is shadowed by "humans" etc. The heaven of mathematical ideas reveals itself similarly in truths. A truth in mathematics might have an independent existence in a Platonic heaven or in the mind. This was the point of divide between intuitionists and classical logicians. The word "there exists" has been the point of contention.

Anyway, this does sound something beyond mathematics. I won't go into much details overe here for the same reason. I'm just remarking what it really is that had issues with so solid a system of logic as we are wont to deal with. Intuitionists had precisely such issues. They said that the abode for mathematical ideas is only in the mind. Personally, I don't know what the philosophy of either means and I will not delve into the philosophical issues of logic. I only wanted to provide a motivation.

Having done that, let's move to some logic now. Let's start with the law of excluded middle. For any proposition p, $p \vee \backsim p$ is a tautology because things are either true or false. In mathematics, things are black and white. According to intuitionists, "true" means provable and false means that the proposition leads to a contradiction or that the negation of the proposition can be proved. Based on this definition of true and false, in 1907, the PhD thesis of a student, L. E. J. Brouwer, to the university of Amsterdam, argued for the invalidity of the law of excluded middle. Brouwer insisted that the law of excluded middle had been generalised without consulting all cases. One way to prove that $p \vee \backsim p$ is valid is to prove it for all theorems, which, for obvious reasons, is impractical. On the other hand, only one example suffices to invalidate any law. Such an example presented by Brouwer goes thus: consider the case where it is purported that there are an infinite number of pairs of twin primes. It is not known if there are an infinite number of such primes (Wells 1986, p. 41; Shanks 1993, p. 30) but it seems almost certain to be true. While Hardy and Wright (1979, p. 5) note that "the evidence, when examined in detail, appears to justify the conjecture," and Shanks (1993, p. 219) states even more strongly, "the evidence is overwhelming," Hardy and Wright also note that the proof or disproof of conjectures of this type "is at present beyond the resources of mathematics." Now, assume $x, y \in \mathbb{Z}^+$ and $q(x)$ abbreviates the property "there is a $y > x$ such that both y and $y + 2$ are prime numbers". Then, we have no general method for deciding whether $q(x)$ is true or false for arbitrary x, so $\forall x(q(x) \vee \backsim q(x))$ cannot be asserted in the present state of our knowledge. There is a third "unknown" which invalidates this law. The example mentioned rests on the account that the twin prime conjecture has not been proven and the truth or falsity of the conjecture remains, till date, unknown. Fermat's last theorem was another example Brouwer presented. The presentation of a correct proof of Fermat's last theorem does not imply the invalidity of the law of the excluded middle, according to intuitionists. If something is true, it must have a proof. If it isn't, it leads to a contradiction. If there's a proposition without a proof or that the proof of its negation does not exist, then that proposition has an unknown value. Basically, what intuitionists deny is the knowledge of the fact that all mathematical problems have solutions. In fact, this is where the law of excluded middle divides them. With the answer to the continuum hypothesis, the position of an intuitionist is only strengthened.

$a \wedge b$

a/b	0	1/2	1
0	0	0	0
1/2	0	1/2	1/2
1	0	1/2	1

$a \vee b$

a/b	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1
1	1	1	1

$a \Rightarrow b$

a/b	0	1/2	1
0	1	1	1
1/2	0	1	1
1	0	1/2	1

a	$\backsim a$
0	1
1/2	0
1	0

Notice that $1/2 \vee \backsim 1/2 = 1/2 \vee (1/2 \Rightarrow 0) = 1/2 \vee 0 = 1/2$ falsifies the law of excluded middle.

To set the record straight, please note that it is because of an unknown truth value that the law of excluded middle fails to hold. The law of non-contradiction, that is either p holds or $\backsim p$ holds and both cannot hold at the same time, is still valid in intuitionistic logic.

In set theory, this says that for sets $A \subset X$, $A \cup A^c \subseteq X$. The Zermelo-Fraenkel set theory with the axiom of choice is replace by a milder yet embeddable Intuitionistic FZ theory. The axiom of choice is excluded because the axiom of choice implies the law of excluded middle. How? For every set $A \subset X$, we can construct a complement such that $A \cup A^c = X$. More set relations can be derived from the structure of intuitionistic logic, mentioned at the end.

According to David Hilbert, taking away the law of excluded middle is like taking away the fists of a boxer. Let's look at other consequences of the denial of the law of excluded middle. Classically, all proofs whatsoever of the law of excluded middle follow from double negation and conversely. Consider $\backsim\backsim p$ holds and p does not. From p, we can have $p \vee \backsim p$. Note that this is also propositional weakening. According to assumption, if $\backsim p$ holds, then $\backsim\backsim p$ does not hold, which contradicts the assumption. Hence $\tilde{p}$ is impossible. That leaves us with p, which, we've already assumed, does not hold. Hence, the initial assumption is invalidated, which means both either hold or do not hold i.e. their truth values are the same. Hence $\backsim\backsim p = p$. Conversely, any proof of the double negation will rely on the law of excluded middle. Thus, intuitionists reject the law of double negation, as well. As a pun, it is usually said that a classical thinker is one who cannot say anything positive! For instance, it is often inconvenient for construcitivists to say that a set is non-empty (a set which does not have no elements). Instead, they prefer the term "a set with at least one element".

The most interesting propositional connective is the implication, without which the whole of mathematics might be pretty useless. Classically, $A \implies B$

is true if A is false or if B is true. In intuitionistic logic, this cannot be used because the classical disjunction is used; moreover, it assumes that the truth values of A and B are known before one can settle the status of $A \implies B$. Consider $A =$"there occur nineteen consecutive 7's in the decimal expansion of π", and $B =$"there occur eighteen consecutive 7's in the decimal expansion of π". Then $\backsim A \vee B$ does not hold constructively but the implication, $A \implies B$ is obviously correct. In intuitionistic logic, implication is modified to say that $A \implies B$ is true if there is an algorithm or a method by which a proof of B can be deduced from the proof of A.

As already mentioned, the point of contention between intuitionistic logic and classical logic is in the word "exists". Let us assume that we have an object x with a property $Q(x)$. In classical logic, this means that there does not exist an object x such that $\backsim Q(\dot{x})$. The proof of the fomer would require showing that such an object exists whereas the latter would mean that $\backsim Q(x)$ leads to a contradiction and proofs by contradiction are also not allowed in intuitionistic logic. An implication of this is that objects have to be constructed in existence proofs. Technically, this means that objects have to be constructed or invented, which is what an intuitionist insists on, instead of being discovered. This follows from the intuitionist's line of philosophy. Rigorously, suppose $p \implies q$ is a theorem. The technique of proof by contradiction starts by assuming that the theorem is incorrectfrom which we can derive a contradiction $r \wedge \backsim r$ (intuitionists accept that this is a contradiction!). So, basically, $\backsim (p \Rightarrow q) \implies (r \wedge \backsim r)$. We've assumed that the hypothesis is false and reached a false conclusion, implying that the implication is true. Hence $\backsim (p \implies q)$ is false or $p \implies q$ is true. In the last line, double negation has to be followed for existence proofs, which is why proof by contradiction is not allowed.

To reiterate, intuitionists insist on constructing objects instead of assuming their non-existence and deriving a contradiction. The axiomatics for each case happen to be the same except for the law of excluded middle. The mathematical entities constructed in intuitionisitc logic are the same as those constructed in classical logic. For instance, the graph of a circle is described in classical logic using the equation $x^2 + y^2 = r^2$ whereas in order to construct a circle, one would bother with parametrization $(\cos t, \sin t)$. This might seem a trivial beating about the bush and has been viewed so. This adds to the remark that intuitionistic logic has only added difficulties for itself but classical logic stands as it is. Intuitionistic mathematics is not viewed as alternative to mathematics but only asks to look at mathematics from a different vantage. The objects being discussed happen to be the same but the notion of proof differs dramatically but that is not all; some theorems of classical logic do not hold in intuitionistic logic. For instance, the law of trichotomy is clearly a result of the law of excluded middle. Intuitionists, on the other hand, have a construction for determining whether a real number is zero or non-zero and that is all. Also, the intermediate value theorem is invalid in constructive logic.

Now perhaps is a good place to state a definition. For any proposition A, let $\vdash_I A$ denote the fact that A is provable in intuitionisitc logic. Also, let $\models_I A$

denote the fact that A is intuitionistic tautology. The proof system I for the intuitionistic logic has hence to be such that $\nvdash_I (\backsim \forall x \backsim A(x)) \implies \exists x A(x)$ and similarly for the tautology. Considering the knowledge of negation, from the proof of A, the proof of $\backsim\backsim A$ is provable. That is, $A \implies \backsim\backsim A$. This is because if A is provable, then $\backsim A$ leads to a contradiction, which implies that $\backsim\backsim A$ is provable. However, if $\backsim\backsim A$ is provable, then one can't go to A by the same reasoning. Hence, $\nvdash_I \backsim\backsim A \implies A$.

The completeness theorem for Intuitionistic logic says that if any proposition A is provable if and only if it is a tautology. That is, $\vdash_I A \iff \models_I A$.

Remember when Georg Cantor said that the set is the many which allows itself to be thought of as one? Well, intuitionists have issues with that as well. Brouwer said that the law of excluded middle was abstracted from finite situations and then extended to infinite situations without justification. For instance, the Goldbach conjecture can be verified "empirically" but no proof of it exists so far. According to intuitionists, an infinite set cannot be made into a whole because it is in a state of constant formation. Hence, they have talks about potential infinity against actual infinity. For instance, the natural numbers are constructed as follows: 1 belongs to the set $\mathbb{N}$ and a step function S is such that $S(n) \in \mathbb{N}$ implies $S(n+1) \in \mathbb{N}$. This way, the set of natural numbers is constructed whereas in classical logic, the natural numbers are assumed to exist. Note that this does not mean that intuitionists deny the existence of infinite natural numbers. Since the notion of a complete infinite set is absent, intuitionist deny the construction of $\mathcal{P}(\mathbb{N})$ because they don't have an algorithm for it.

Let's look at an example of a theorem with constructive and non-constructive proofs.

Theorem 1 *There are irrational numbers x and y such that x^y is a rational number.*

Proof. (Non-constructive)

By the law of excluded middle, $\sqrt{2}^{\sqrt{2}}$ is a either rational or irrational. If $\sqrt{2}^{\sqrt{2}}$ is rational, then we're done. If the number is irrational, then let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ from which we get $x^y = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$ ■

This proof is non-constructive since it does not present the number itself but only runs because of the lack of negative evidence after an exhaustive search by applying the law of excluded middle. Now for a constructive proof

Proof. We need to construct such a rational number. For that, we take $x = \sqrt{2}$ and $y = \log_2 9$ to get $\sqrt{2}^{\log_2 9} = 2^{1/2 * \log_2 9}$

$= 2^{\log_2 3} = 3.$ ■

There are constructive proofs for the irrationality of real numbers. They basically show that there is a finite difference between the irrational number and any rational number.

Also, Cantor's famous diagonal argument is a *reductio ad absurdum* and can be written constructively as follows: "Let $\{a_n\}$ be a sequence of real numbers. Let x_0 and y_0 be real numbers, $x_0 < y_0$. Then there exists a real number x with $x_0 \leq x \leq y_0$ and $x \neq a_n$ for any $n \in \mathbb{Z}^+$. There is a seperate real analysis called constructive analysis based on these principles. The proofs are rather long.

To repeat, the law of excluded middle is basically the point of divide between intuitionists and classical mathematicians, which stems from the fact that they believe the word "exists" has a different meaning. In other words, intuitionistic logic demands positive evidence, while classical logic is happy with lack of negative evidence. This is closer to the spirit of science. If A holds, then there is evidence to support it. If $\backsim A$ holds, then assuming A to hold would lead to a contradiction. If $\backsim\backsim A$, then A cannot be falsified because it is contradictory to assume $\backsim A$. A could be true, for all we know but current state of knowledge suggests otherwise. For example, "there is a particle that does not interact with anything in the universe" will be considered true by a classical reasoner but an intuitionist will be very suspicious with it, firstly, because of the double negation and, secondly, because the lack of negative evidence is not sufficient. This is potentially true and not falsifiable. That's the trouble with classical reasoning.

"I didn't ask you to not do that" does not necessarily mean that you ought to have done it!

In the above discussion, all the discrepancies between classical and intuitionistic logic stem from the rejection of the law of excluded middle. On the other hand, once the law of excluded middle is included in the axioms for intuitionistic logic, one has for himself a system of classical logic. Here are examples of classical tautologies that are not intuitionistic tautologies:

1. $\backsim A \vee A$
2. $\backsim\backsim A \implies A$ (if something is not false, then we can't say that it is true)
3. $(A \implies B) \implies (\backsim A \vee B)$
4. $\backsim (A \wedge B) \implies (\backsim A \vee \backsim B)$
5. $(\backsim A \implies B) \implies (\backsim B \implies A)$
6. $(\backsim A \implies \backsim B) \implies (B \implies A)$

Since intuitionistic logic is a weakening of classical logic, we have $\vdash_I A \implies \vdash_C A$. Similarly, $\models_I A \implies \models_C A$. All theorems of intuitionistic logic hold classically. This is one way of going to classical logic from intuitionistic logic. To go the other way round and hence to make sense of intuitionistic logc in classical logic, we have the

Theorem 2 *(Glivenko theorem)* $\vdash_C A \iff \vdash_I \backsim\backsim A$.

An implication of this is Tarski's theorem, which states that $\models_C \backsim\backsim A \iff \models_I A$. Godel proved that $\vdash_C (A \implies \backsim B) \iff \vdash_I (A \implies \backsim B)$, from which

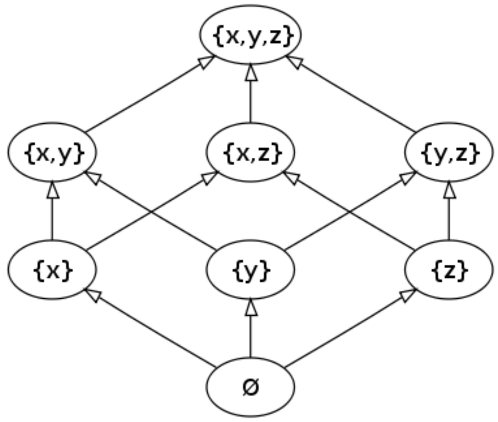

Figure 1: The Hasse diagram of the set of all subsets of a three-element set $\{x, y, z\}$, ordered by inclusion.

we can get $\models_C (A \implies \backsim B) \iff \models_I (A \implies \backsim B)$. Similarly, propositions involving $\backsim$ and $\wedge$ only hold in intuitionisitc logic $\iff$ such a proposition holds in classical logic.

Let us move to a more rigorous study of this logic. Before that, I'll start by being rigorous with Classical Logic, so that we don't get lost with the formalisation of intuitionistic logic. Before that, some lattice theory. A Boolean algebra is basically $(\mathbb{Z}_2, +.)$ but this is an algebra, not a structure. A different way to approach the problem is via lattices.

Definition 3 *Let $\leq$ be a binary relation. A **partial order** is a binary relation $\leq$ over a set S if $\forall a, b, c \in S$*

- $a \leq a$ *($\leq$ is reflexive)*
- $a \leq b$ *and* $b \leq a \implies a = b$ *($\leq$ is antisymmetric)*
- $a \leq b$ *and* $b \leq c \implies a \leq c$ *($\leq$ is transitive)*

Definition 4 *A **lattice** $(\mathcal{L}, \leq)$ is a partially ordered set in which the meet/infimum/greatest lower bound and join/supremum/least upper bound of any two elements is defined.*

For $a, b \in \mathcal{L}$, $\inf\{a, b\} = a \wedge b$ whereas $\sup\{a, b\} = a \vee b$

These operations are idempotent, commutative, associative and satisfy the absorption law.

Proof. $a \wedge a = \inf\{a, a\} = a$ and $a \vee a = \sup\{a, a\} = a$

$a \wedge b = \inf\{a, b\} = \inf\{b, a\} = b \wedge a$ whereas $a \vee b = \sup\{a, b\} = \sup\{b, a\} = b \vee a$

$(a \wedge b) \wedge c = \inf\{c, \inf\{a, b\}\} = \inf\{\inf\{b, c\}, a\}\} = a \wedge (b \wedge c)$ and similarly for supremum

Finally, $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ ■

We can also define $a \leq b \iff a \wedge b = a$ whereas $a \leq b \iff a \vee b = b$ The following identities hold in any lattice

Theorem 5 $a \wedge b \leq b$ *and* $a \wedge b \leq a$

Proof. A direct consequence of definition ■

Theorem 6 $b \leq a \vee b$ *and* $a \leq a \vee b$

Proof. A direct consequence of definition ■

Theorem 7 $a = b \wedge a \iff a \vee b = b$

Proof. Use the fact that partial order is anitsymmetric then apply definitions of supremum and infimum ■

Theorem 8 $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

Theorem 9 $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Theorem 10 $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \leq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$

Theorem 11 $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee (a \wedge c))$

A good example is the usual set notation of intersection, union and "contained in" for a lattice where the set $\mathcal{L}$ is $\mathcal{P}(X)$ for any set X. The positive integers in their usual order form a lattice, under the operations of "min" and "max". 1 is bottom; there is no top. Hence not every lattice is bounded. The natural numbers also form a lattice under the operations of taking the greatest common divisor and least common multiple, with divisibility as the order relation: $a \leq b$ if $a|b$. 1 is bottom; 0 is top – this lattice is bounded

Now, in the partial order, let us identity 0 as the least element and 1 as the maximal element. Then, clearly we have $a \vee 0 = \sup\{a, 0\} = 0$ and $a \wedge 1 = \inf\{a, 1\} = a$. This is a bounded lattice. The distributive law does not hold for every lattice, as can be seen from the above properties but we can impose the distributive law, as well, as an axiom. Adding an additional operation of complement i.e. for every a, there should exist a b such that $a \vee b = 1$ and $a \wedge b = 0$, we get a complemented bounded lattice. These complements are not unique, unless the lattice satisfies the distributive law. We can denote this element as $\backsim a$, if the lattice is distributive. This is a Boolean algebra and obeys all the usual rules of classical logic we know. For the record, there are other examples of Boolean algebras; for instance, topological spaces. The following properties are satisfied in a Boolean Lattice:

Theorem 12 *If* $x \vee o = x$ *for all* x*, then* $o = 0$

Proof. $0 = 0 \vee o = o \vee 0 = o$ ■

Theorem 13 $x \vee x = x$

Proof. $x \vee x = (x \vee x) \wedge 1 = (x \vee x) \wedge (x \vee \backsim x) = x \vee (x \wedge \backsim x) = x \vee 0 = x$ ■

Theorem 14 $x \vee 1 = 1$

Proof. $x \vee 1 = (x \vee 1) \wedge 1 = 1 \wedge (x \vee 1) = (x \vee \backsim x) \wedge (x \vee 1) = x \vee (\backsim x \wedge 1) = x \vee \backsim x = 1$ ■

Theorem 15 $x \vee (x \wedge y) = x$

Proof. $x \vee (x \wedge y) = (x \wedge 1) \vee (x \wedge y) = x \wedge (1 \vee y) = x \wedge (y \vee 1) = x \wedge 1 = x$ ■

Theorem 16 *If* $x \vee x_n = 1$ *and* $x \wedge x_n = 0$, *then* $x_n = \backsim x$

Proof. $x_n = x_n \wedge 1 = 1 \wedge x_n = (x \vee \backsim x) \wedge x_n = x_n \wedge (x \vee \backsim x) = (x_n \wedge x) \vee (x_n \wedge \backsim x)$
$= (x \wedge x_n) \vee (\backsim x \wedge x_n) = 0 \vee (\backsim x \wedge x_n) = (x \wedge \backsim x) \vee (\backsim x \wedge xn) = (\backsim x \wedge x) \vee (\backsim x \wedge x_n)$
$= \backsim x \wedge (x \vee x_n) = \backsim x \wedge 1 = \backsim x$ ■

Theorem 17 $\backsim\backsim x = x$

Proof. $\backsim x \vee x = x \vee \backsim x = 1$ and $\backsim x \wedge x = x \wedge \backsim x = 0$ hence $x = \backsim\backsim x$ ■

Theorem 18 $x \vee (\backsim x \vee y) = 1$

Proof. $x \vee (\backsim x \vee y) = (x \vee (\backsim x \vee y)) \vee 1 = 1 \wedge (x \vee (\backsim x \vee y)) = (x \vee \backsim x) \wedge (x \vee (\backsim x \vee y))$
$= x \vee (\backsim x \wedge (\neg x \vee y)) = x \vee \backsim x = 1$ ■

Theorem 19 $(x \vee y) \vee (\backsim x \wedge \backsim y) = 1$

Proof. $(x \vee y) \vee (\backsim x \wedge \backsim y) = ((x \vee y) \vee \backsim x) \wedge ((x \vee y) \vee \backsim y) = (\backsim x \vee (x \vee y)) \wedge (\backsim y \vee (y \vee x))$
$= (\backsim x \vee (\backsim\backsim x \vee y)) \wedge (\backsim y \vee (\backsim\backsim y \vee x)) = 1 \wedge 1 = 1$ ■

Theorem 20 $(x \vee y) \wedge (\backsim x \wedge \backsim y) = 0$

Proof. $(x \vee y) \wedge (\backsim x \wedge \backsim y) = (\backsim x \wedge \backsim y) \wedge (x \vee y) = ((\backsim x \wedge \backsim y) \wedge x) \vee ((\backsim x \wedge \backsim y) \wedge y)$
$= (x \wedge (\backsim x \wedge \backsim y)) \vee (y \wedge (\backsim y \wedge \backsim x)) = 0 \vee 0 = 0$ ■

Theorem 21 $\backsim (x \vee y) = \backsim x \wedge \backsim y$

Theorem 22 $(x \vee (y \vee z)) \vee \backsim x = 1$

Proof. $(x \vee (y \vee z)) \vee \backsim x = \backsim x \vee (x \vee (y \vee z)) = \backsim x \vee (\backsim\backsim x \vee (y \vee z)) = 1$ ■

Theorem 23 $y \wedge (x \vee (y \vee z)) = y$

Proof. $y \wedge (x \vee (y \vee z)) = (y \wedge x) \vee (y \wedge (y \vee z)) = (y \wedge x) \vee y = y \vee (y \wedge x) = y$ ■

Theorem 24 $(x \vee (y \vee z)) \vee \backsim y = 1$

Proof. $(x \vee (y \vee z)) \vee \backsim y = \backsim y \vee (x \vee (y \vee z)) = (\backsim y \vee (x \vee (y \vee z))) \wedge 1 = 1 \wedge (\backsim y \vee (x \vee (y \vee z)))$
$= (y \vee \backsim y) \wedge (\backsim y \vee (x \vee (y \vee z))) = (\backsim y \vee y) \wedge (\backsim y \vee (x \vee (y \vee z))) = \backsim y \vee (y \wedge (x \vee (y \vee z)))$
$= \backsim y \vee y = y \vee \backsim y = 1$ ■

Theorem 25 $(x \vee (y \vee z)) \vee \backsim z = 1$

Proof. $(x \vee (y \vee z)) \vee \backsim z = (x \vee (z \vee y)) \vee \backsim z = 1$ ■

Theorem 26 $\backsim ((x \vee y) \vee z) \wedge x = 0$

Proof. $\backsim ((x \vee y) \vee z) \wedge x = (\backsim (x \vee y) \wedge \backsim z) \wedge x = ((\backsim x \wedge \backsim y) \wedge \backsim z) \wedge x = x \wedge ((\backsim x \wedge \backsim y) \wedge \backsim z)$
$= (x \wedge ((\backsim x \wedge \backsim y) \wedge \backsim z)) \vee 0 = 0 \vee (x \wedge ((\backsim x \wedge \backsim y) \wedge \backsim z))$
$= (x \wedge \backsim x) \vee (x \wedge ((\backsim x \wedge \backsim y) \wedge \backsim z)) = x \wedge (\backsim x \vee ((\backsim x \wedge \backsim y) \wedge \backsim z))$
$= x \wedge (\backsim x \vee (\backsim z \wedge (\backsim x \wedge \backsim y))) = x \wedge \backsim x = 0$ ■

Theorem 27 $\backsim ((x \vee y) \vee z) \wedge y = 0$

Proof. $\backsim ((x \vee y) \vee z) \wedge y = \backsim ((y \vee x) \wedge z) \wedge y = 0$ ■

Theorem 28 $\backsim ((x \vee y) \vee z) \wedge z = 0$

Proof. $((x \vee y) \vee z) \wedge z = (\backsim (x \vee y) \wedge \backsim z) \wedge z = z \wedge (\backsim (x \vee y) \wedge \backsim z) = z \wedge (\backsim z \wedge \backsim (x \vee y)) = 0$ ■

Theorem 29 $(x \vee (y \vee z)) \vee \backsim ((x \vee y) \vee z) = 1$

Proof. $(x \vee (y \vee z)) \vee \backsim ((x \vee y) \vee z) = (x \vee (y \vee z)) \vee (\backsim (x \vee y) \wedge \backsim z) = (x \vee (y \vee z)) \vee ((\backsim x \wedge \backsim y) \wedge \neg z)$
$= ((x \vee (y \vee z)) \vee (\backsim x \wedge \backsim y)) \wedge ((x \vee (y \vee z)) \vee \backsim z)$
$= (((x \vee (y \vee z)) \vee \backsim x) \wedge ((x \vee (y \vee z)) \vee \backsim y)) \wedge ((x \vee (y \vee z)) \vee \backsim z)$
$= (1 \wedge 1) \wedge 1 = 1$ ■

Theorem 30 $(x \vee (y \vee z)) \wedge \backsim ((x \vee y) \vee z) = 0$

Proof. $(x \vee (y \vee z)) \wedge \backsim ((x \vee y) \vee z)$
$= \backsim ((x \vee y) \vee z) \wedge (x \vee (y \vee z)) = (\backsim ((x \vee y) \vee z) \wedge x) \vee (\backsim ((x \vee y) \vee z) \wedge (y \vee z))$
$= (\backsim ((x \vee y) \vee z) \wedge x) \vee ((\backsim ((x \vee y) \vee z) \wedge y) \vee (\backsim ((x \vee y) \vee z) \wedge z))$
$= (0 \vee 0) \vee 0 = 0$ ■

If this is not enough, we can always have a partition with an equivalence relation R such that pRq if $p \Leftrightarrow q$. We can thus partition this logic and thus theorems. The quotient obtained has a special name (Lindenbaum-Tarski algebra). This is classical sentential/propositional/sequential logic.

There's one particular important thing that needs to be noted:

Let $(\mathcal{L}, \wedge, \vee, \leq)$ be a Uniquely Complemented Lattice. Then the following are equivalent:

1. $\forall a, b \in \mathcal{L} : \backsim a \vee \backsim b = \backsim (a \wedge b)$
2. $\forall a, b \in \mathcal{L} : \backsim a \vee \backsim b = \backsim (a \vee b)$
3. $\forall a, b \in \mathcal{L} : a \leq b \iff \backsim b \leq \backsim a$
4. $(\mathcal{L}, \wedge, \vee, \leq)$ is a Distributive Lattice.

We can define implication $a \implies b$ as an operation to yield an element of the lattice, equivalent to $\backsim a \vee b$.

Now let's move to the intuitionistic part of logic. Let's say we have a bounded lattice $\mathcal{L}$. Now, we can define an implication operation $a \implies b$ as the maximal element of the set $T(a,b) := \{x \in \mathcal{L} | a \wedge x \leq b\}$. Hence, we have $a \wedge x \leq b \iff x \leq (a \implies b)$. This maximal, if exists, is unique. Note that the implication is now an operation and $a \implies b \in \mathcal{L}$. c is called the relative pseudo-complement of a with respect to b and this is denoted by $a \implies b$. The pseudo-complement of a is defined as $\backsim a := a \implies 0$. Using this definition and the definition of a lattice, we get the following results.

Theorem 31 $a \implies a$

Proof. $T(a,a) = \mathcal{L}$ hence $\max \{T(a,a)\} = 1$. Therefore, $1 = a \implies a$
$\blacksquare$

Theorem 32 $b \leq (a \implies b)$

Proof. This holds since $a \wedge b \leq b$ $\blacksquare$

Theorem 33 $(a \implies 1) = 1$

Proof. $T(a,1) = \{x \in \mathcal{L} | a \wedge x \leq 1\} = \mathcal{L}$ $\blacksquare$

Theorem 34 $a \leq b \iff (a \implies b) = 1$

Theorem 35 $(1 \implies a) = 1$ *then* $a = 1$

Theorem 36 $a \wedge (a \implies b) = a \wedge b$

Proof. Since $b \leq (a \implies b)$, we have $a \wedge b \leq a \wedge (a \implies b)$. Conversely, by definition, $a \wedge (a \implies b) \leq a$ as well as $a \wedge (a \implies b) \leq a \wedge b$. Using the fact that $\leq$ is antisymmetric, our proof is complete. $\blacksquare$

Theorem 37 $a = (1 \implies a)$

Proof. From the previous theorem, we have $1 \wedge (1 \implies a) = a \wedge 1$ or $(a \implies 1) = a$ ■

Theorem 38 *$a \leq b$, then $(c \implies a) \leq (c \implies b)$*

Proof. $c \wedge (c \implies a) = c \wedge a \leq a \leq b \leq (c \implies b)$ ■

Theorem 39 *$a \leq b$, then $(b \implies c) \leq (a \implies c)$*

Proof. $a \wedge (b \implies c) \leq b \wedge (b \implies c) = b \wedge c \leq c$ ■

Theorem 40 $a \implies (b \implies c) = (a \wedge b) \implies c = (a \implies b) \implies (a \implies c)$

Theorem 41 *Distributive law holds*

Theorem 42 $\backsim 1 = 0$

Theorem 43 $\backsim a = 1 \iff a = 0$

Theorem 44 $a \leq \backsim\backsim a$

Theorem 45 $\backsim\backsim\backsim a = \backsim a$

Theorem 46 $\backsim a \leq (a \implies b)$

Theorem 47 $(a \implies b) \wedge (a \implies \backsim b) = \backsim a$

Theorem 48 $(a \implies b) \leq (\backsim b \implies \backsim a)$

Theorem 49 $\backsim a \vee b \leq (a \implies b)$

Theorem 50 *A Heyting Lattice is a Boolean Lattice* $\iff \backsim\backsim a \leq a$

Theorem 51 $[\backsim\backsim a \leq a] \iff [\backsim a \vee b = (a \implies b)]$

In summary, in intuitionistic logic, one constructs objects. This corresponds to being able to find a way to construct an object or by formulating an algorithm. As far as mathematics today is concerned, better grounds can be achieved using a computer. A set can be a data structure and the use of intuitionistic logic can yield interesting data. Alternatively, constructions can be replaced by continuous functions and data structures by topological spaces.

Topos theory is an interesting application of Intutionistic Logic. I don't have the time to describe it completely, because first I'd have to go through category theory and then to Topos theory. I plan to do that in my next lecture. For now, I have a good non-technical book on the subject from the point of view of an application. It's Smolin Lee's Three Roads to Quantum Gravity. Two roads are well known viz. String Theory and Loop Quantum Gravity. The third approach involves Topos Theory, which derives from Intuitionistic Logic.

Some open problems are the development of foundations of manifolds and Riemannian geometry in University of Canterbury by Prof. D. S. Bridges